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Scaling behaviour of the correlation length for the two-point correlation function in the random field Ising chain

Adrian Lange and Robin Stinchcombe
Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

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Abstract. We study the general behaviour of the correlation length $\xi(kT, h)$ for the two-point correlation function of the local fields in an Ising chain with binary distributed fields. At zero field it is shown that ξ is the same as the zero-field correlation length for the spin–spin correlation function. For the field-dominated behaviour of ξ we find an exponent for the power-law divergence which is smaller than the exponent for the spin–spin correlation length. The entire behaviour of the correlation length can be described by a single crossover scaling function involving the new critical exponent.

1. Introduction

Following the discovery [1] of universal scaling behaviour in a simple bifurcation route to chaos, more and more detailed studies have emphasized the rich, typically multifractal properties of many simple iterative maps [2]. This development has occurred in parallel with the description of scaling properties near continuous phase transitions. Here the crucial understanding of critical behaviour and, in particular, of scaling and universality comes from the renormalization group [3], which is essentially a map of system parameters occurring through a length scale change. A particularly challenging class of systems exhibiting criticality at a continuous phase transition is the class of random spin models, especially the frustrated ones of which the best known examples are the spin glass [4] and the random-field models of Ising type [5]. These have been much explored both because of their rich and sometimes surprising behaviour and because of their relevance to real systems ranging from physical ones (like dilute transition metal alloys or dilute antiferromagnets in a field) to biological ones (such as neural networks).

At the most basic level, their analogous scaling behaviours have long provided a link between the iterative maps and the continuous phase transitions. On the other hand, multifractal characteristics, common in iterative maps, do not show up in the behaviour of simple non-random phase transitions where k-spin correlation functions for different k are rather trivially linked by ‘gap scaling’ [6]. Nevertheless, multifractal features do commonly occur in random systems such as dilute spin systems and other processes on the percolation network [7]. So it is of interest to investigate the relationship between iterative maps and random systems. That is one aim of the present paper, where the random-field Ising model is investigated.

Another reason for attempting to view the random spin systems via an iterative map is because this can be a very effective way of approaching them. In one dimension ($d = 1$)

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the non-random Ising model can be trivially solved by using a (multiplication of a) transfer matrix to achieve the sum over spin configurations involved in its partition function [8]. But even in $d = 1$ this procedure is very difficult for the random Ising models, where both a spin configuration sum and a quenched average over randomness is required [9]. An elegant alternative method is provided by a mapping method, introduced by Ruján *et al* [10, 11] and applied by Behn *et al* [12–15] to one-dimensional Ising spin glass and neural network models. Here the mapped variable at each stage n of the mapping is a 'local field' including the exact fluctuating effects of all spins to the left of site n in the actual spin chain. This technique is used on the random-field Ising chain (RFI) in the present paper. There are indications that in the limit of zero temperature T this method is related to an algorithm [16] for constructing the RFI ground state.

For most experiments, spin–spin correlation functions are most relevant, since they are measured by standard probes such as linear response to an adiabatic or isothermal applied field, or scattering of neutrons or electromagnetic waves [17]. Nevertheless, other correlation functions can exhibit new features and may occur naturally in iterative maps. An example is the correlation between the value of a mapped variable at different stages of the mapping. This paper focuses on such a correlation, corresponding to that between 'local fields' at different stages of the RFI mapping or, equivalent, between local fields at different sites in the random field Ising chain.

It is found that the associated scaling behaviour, which occurs at low field and low temperature, can be characterized by a crossover form involving an apparently new critical exponent.

The layout of the paper is as follows. In section 2 we describe the model and the mapping which leads to the two-point correlation function. In section 3 the numerical results for the correlation lengths are presented and compared with the analytical forms. Finally, we summarize our results in section 4.

2. Model

We consider a finite chain of N spins with ferromagnetic exchange ($J > 0$) in a quenched random field $\{h_n\}$

$$H = -J\sum_{n=1}^{N} s_n s_{n+1} - \sum_{n=1}^{N} h_n s_n \tag{1}$$

with $s_n = \pm 1$ and $s_{N+1} = 0$. The random fields h_n with zero mean value are independent binary random variables, each taking the two values $h_\sigma = \sigma h$, $\sigma = \pm 1$, with the same probability $\frac{1}{2}$. Because of the Markov character, the exact calculation of the partition function for (1) can be reduced to the problem of *one* spin in an effective local random field x_n [10, 11]

$$Z^N = \sum_{\{s_n\}} \exp\left[\beta\left(\sum_{n=1}^{N-1} J s_n s_{n+1} + \sum_{n=1}^{N} h_n s_n\right)\right] = \sum_{s_N} \exp\left[\beta\left(x_N s_N + \sum_{n=1}^{N-1} B(x_n)\right)\right] \tag{2}$$

by summing configurations starting with the left-most spin in the chain. β is the inverse temperature $1/kT$ and $B(x_n)$ is the function

$$B(x_n) = \frac{1}{2\beta}\ln\left[\cosh\beta(x_n + J)\cosh\beta(x_n - J)\right]\ . \tag{3}$$

The effective local random field x_n acting on spin s_n is governed by the discrete stochastic mapping [11]

$$x_0 = 0 \qquad x_n = h_n + A(x_{n-1}) \equiv f(h_n, x_{n-1}) \qquad n = 1, \ldots, N \tag{4}$$

where

$$A(x_n) = \frac{1}{2\beta} \ln \frac{\cosh \beta(x_n + J)}{\cosh \beta(x_n - J)} \,. \tag{5}$$

This new local field x_n is a superposition of the (external) field h_n and a field $A(x_{n-1})$ which contains the effect of all spins left of n.

The corresponding probability density $p_n(x)$ for the mapping (4) is governed by the Frobenius–Perron equation. Its fixed point gives the invariant measure of the local field x_n. For non-zero temperature $A(x)$ is infinitely differentiable where $|\partial_x A(x)| < 1$ holds. Therefore the mapping (4) is *non-chaotic* but generates for a discrete driving process an *uncountable* number of states and a fractal attractor. Since $A(x)$ is nonlinear, the mapping has infinitely many scales and consequently the invariant measure forms a *multifractal*. Its support is either of the topology of a Cantor set or a continuum, depending on the physical parameters [12–14, 18–21]. Particularly the scaling behaviour of the invariant measure at the boundaries of the support shows a notable richness [15].

These drastic changes of the measure and its support are characterized by the behaviour of the generalized fractal dimension D_q [22]. They show as a function of the physical parameters continuous as well as discontinuous transitions which are similar to those of order parameters in phase transitions [12, 13, 18–21]. The multifractal spectrum is closely related to the fluctuations of the free energy of a finite chain [21] and transfers directly to a multifractal distribution of the local magnetization $m_n = \tanh[x_n + A(x'_{n+1})]$, where x'_n contains the effect of all spins right of n.

A different approach is the study of random Ising models by their correlation functions. The few available exact results concern the two-point *connected* correlation function for a diluted symmetric exponential distribution at any temperature [23, 24] or the binary distribution at any temperature [25] and zero temperature [26, 27], respectively.

In this paper we focus on the general behaviour of the correlation length $\xi(kT, h)$ for the two-point correlation function of the local fields:

$$\langle x_{1+s} x_1 \rangle = 2^{-s} \sum_{\sigma_s} \sum_{\sigma_{s-1}} \cdots \sum_{\sigma_1} f(h_{\sigma_s}, f(h_{\sigma_{s-1}}, \ldots, f(h_{\sigma_1}, x_1) \cdots)) x_1 \,. \tag{6}$$

The points are separated by a fixed number s of iterations, which is equivalent to a fixed spatial separation in the original system (1). $\langle \cdots \rangle$ means an average over all configurations of $\{h_1, \ldots, h_s\}$.

From scaling arguments [6] it is expected that the correlations have an asymptotic decay of the form

$$\langle x_{1+s} x_1 \rangle \sim s^{1-\eta} \exp\left(\frac{-s}{\xi(kT, h)} \right) \tag{7}$$

for large enough s provided kT and h are such that ξ is large. η is an exponent related to the scaling dimension of the 'operator' x. A check by means of the $\log\langle x_{1+s} x_1 \rangle$–$\log s$ plot suggests that for all the fields the asymptotic behaviour is already attained for $s \gtrsim 5$ for all non-zero temperatures considered, and $s \gtrsim 20$ for zero temperature, respectively, and that $\eta = 1$. Therefore for $kT \neq 0$ we are able to obtain the scaling behaviour calculating $\langle x_{1+s} x_1 \rangle$ as an average over *all* possible paths for each generation s with $5 \leqslant s \leqslant 19$. For $kT = 0$ we average on the basis of 10^6 randomly chosen paths for each generation s with $20 \leqslant s \leqslant 70$.

3. Results

$d = 1$ is the lower critical dimension of the random-field Ising model [28]. So, in one dimension criticality occurs near the zero field-zero temperature point. That is, for small enough h, kT the correlation length ξ is expected to be large. Depending on the relative sizes of h and kT it should cross over between a *field* dominated ($kT = 0$) form and a *temperature* dominated ($h = 0$) one. In zero field the correlation (7) can be evaluated exactly, since putting

$$x_n = \frac{1}{2\beta} \ln(\tanh \Theta_n) \qquad \Phi = \operatorname{arctanh}\left(e^{-2\beta J}\right) \tag{8}$$

the map (4), (5) becomes

$$\Theta_{n+1} = \Theta_n + \Phi. \tag{9}$$

It is then straightforward to show that for $h = 0$ the large s form (7) applies with $\eta = 1$ and

$$\xi(kT, 0) = \frac{1}{2\Phi} = \left[-\ln\left(\tanh \frac{J}{kT}\right)\right]^{-1}. \tag{10}$$

It is interesting that this is the same, for *all* temperatures, as the correlation length occurring in the usual spin–spin correlation function $\langle s_{n+1} s_1 \rangle$ of the zero-field Ising chain [8]. This correlation length shows the following behaviour in the scaling regime of low temperature

$$\xi(kT, 0) \sim \frac{1}{2} \exp\left(\frac{2J}{kT}\right). \tag{11}$$

From the domain scaling theory it is expected [5] that at $kT = 0$ and small non-zero h the field dominated correlation length for the correlation function $\langle s_{n+1} s_1 \rangle$ of the random field Ising chain is

$$\xi_D(0, h) \sim h^{-2} \tag{12}$$

consistent with the result of Grinstein and Mukamel [23]. For the corresponding correlation length $\xi(0, h)$ for $\langle x_{1+s} x_1 \rangle$, we would also expect (at $kT = 0$ and small h) a power-law divergence

$$\xi(0, h) \sim h^{-\alpha} \tag{13}$$

(since there is no known reason, such as marginal dimensionality, or x_s a marginal operator, to expect otherwise). However, even for small h we have found no reason to expect the same exponent as in (12), so the present work can be used to test whether α is 2.

If h and kT are both sufficiently small, then ξ is large, and in this regime, based on equations (10), (11) and (13) the general behaviour of the correlation length should be described by a crossover scaling function g

$$\xi \sim \frac{1}{2} \exp\left(\frac{2J}{kT}\right) g\left[h^{\alpha} \exp\left(\frac{2J}{kT}\right)\right] \tag{14}$$

such that

$$g(y) = \begin{cases} \text{constant} & \text{for } y \to 0 \\ \dfrac{\text{constant}}{y} & \text{for } y \to \infty. \end{cases} \tag{15}$$

For values $kT/J < 10$ figure 1 shows the exact result for $-1/\xi(kT, 0)$ and the numerical results calculated for $h = 0.005$. The agreement is fairly good as long as the correlation

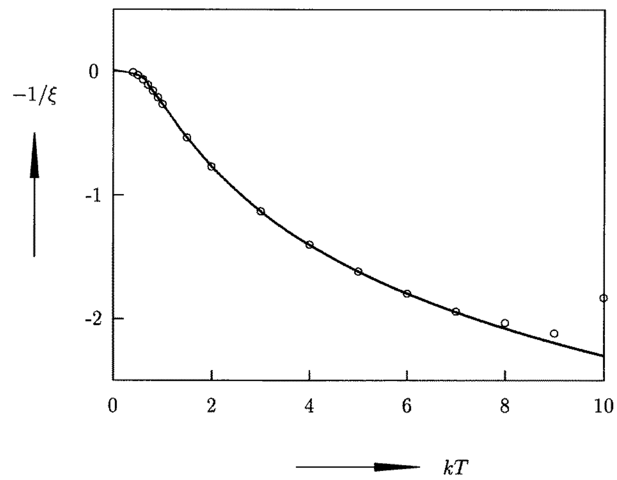

Figure 1. Scaling behaviour of the correlation length in the temperature dominated case ($h = 0$) with $J = 1$. The analytical result (10) for $-1/\xi$ against the temperature kT is denoted by the full curve. The numerical results for $h = 0.005$ are indicated by circles.

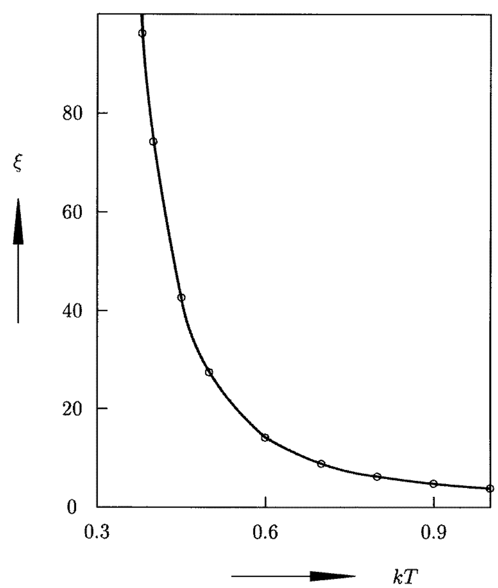

Figure 2. Correlation length ξ in the temperature dominated case ($h = 0$) for low temperatures and $J = 1$. The full curve shows the analytical result (11) whereas the circles indicate the numerical results for $h = 0.001$.

length is larger than $\sim 1/2$ ($kT/J \lesssim 7$). The agreement for the smaller correlation lengths is remarkable since (10) was derived under the assumption of large ξ. In figure 2 the results for the low-temperature behaviour of (10) are shown. The calculated values of ξ for $h = 0.001$ fit very well with (11) over the entire low-temperature range.

The results for the field-dominated case ($kT = 0$) are shown in table 1 and drawn in figure 3. The $\ln 1/\xi$–$\ln h$ plot strongly confirms the power law divergence (13). For the fit we confine ourselves to the four left-most points since we are interested in the *asymptotic* scaling behaviour for non-zero h. From it we determine $\alpha = 1.877 \pm 0.008$ by a least-mean-square fit.

The general behaviour of the correlation length is shown in figure 4. The crossover behaviour from the temperature-controlled one ($\ln[\exp(2J/kT)/2\xi] = 0$) to the field-controlled one ($\ln[\exp(2J/kT)/2\xi]$ increasing linearly) is evidently visible. As expected, the entire behaviour of the correlation length can be described by a single function. The fit

Table 1. The mean value of the inverse correlation length $1/\xi$ in the field dominated case ($kT = 0$) with $J = 1$ for different values of h

h	$1/\xi$
0.09	$0.009\,20 \pm 0.000\,12$
0.11	$0.013\,41 \pm 0.000\,08$
0.13	$0.018\,36 \pm 0.000\,05$
0.16	$0.027\,19 \pm 0.000\,06$
0.19	$0.037\,20 \pm 0.000\,06$
0.21	$0.044\,40 \pm 0.000\,12$
0.23	$0.052\,37 \pm 0.000\,11$
0.26	$0.065\,19 \pm 0.000\,19$
0.29	$0.079\,63 \pm 0.000\,46$

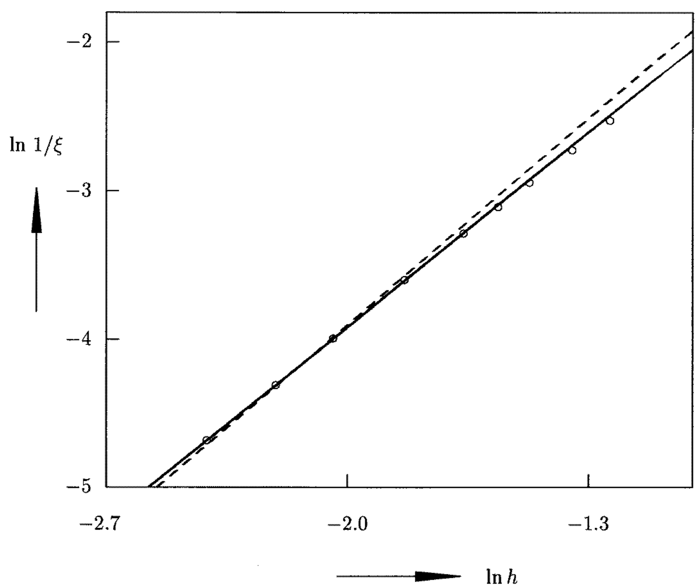

Figure 3. Scaling behaviour of the correlation length in the field dominated case ($kT = 0$) with $J = 1$. The numerical results of $1/\xi$ for $0.09 \leqslant h \leqslant 0.29$ are shown by circles. The full line gives an exponent $\alpha = 1.877$ which fits best for the four left-most points in order to examine the asymptotic scaling behaviour for small non-zero h. For comparison, the broken line shows the fit with $\alpha = 2$.

gives

$$\ln\left[\frac{1}{g(y)}\right] \sim \ln\left[\frac{\exp(2J/kT)}{2\xi}\right] = \ln[1 + 0.07\exp(2.1z - 0.25z^2) + 0.42\exp(z)] \qquad (16)$$

where $z = \ln y = \ln[h^\alpha \exp(2J/kT)]$ and the good data collapse shown in the figure was obtained using $\alpha = 1.877$. Our fit function fulfils (15) since the two interesting limits yield

$$g(y) \sim \begin{cases} 1 & \text{for } y \to 0, \text{ i.e. } z \to -\infty \\ \dfrac{1}{\exp(z)} = \dfrac{1}{y} & \text{for } y \to +\infty, \text{ i.e. } z \to +\infty \end{cases} \qquad (17)$$

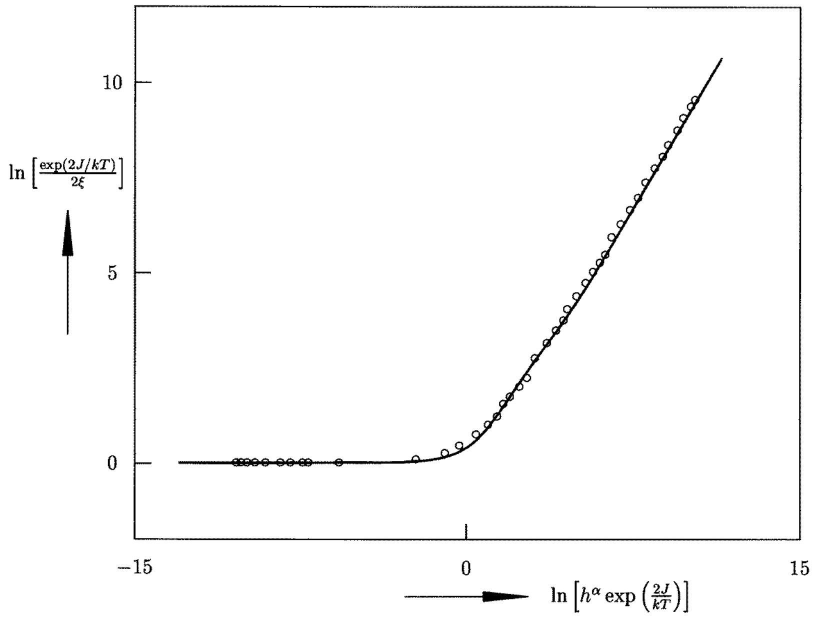

Figure 4. The overall scaling behaviour of the correlation length. The numerical results (○) with $J = 1$ show a crossover behaviour from the temperature-controlled behaviour ($\ln[\exp(2J/kT)/2\xi] = 0$) to the field-controlled one ($\ln[\exp(2J/kT)/2\xi]$ increasing linearly). The calculations were made for h and kT used in figure 2 and for small enough h and kT for which $\xi(h, kT) \geqslant 10$ holds. The entire data set is fitted with $\ln[1 + 0.07 \exp(2.1z - 0.25z^2) + 0.42 \exp(z)]$ where $z = \ln y = \ln[h^\alpha \exp(2J/kT)]$ with $\alpha = 1.877$ (full curve).

4. Discussion

In this paper the general behaviour of the correlation length $\xi(kT, h)$ in the one-dimensional Ising model with binary distributed magnetic field are calculated. The calculations are based on the discrete stochastic mapping (4), (5) for the effective local field acting on the spin at site n.

The correlation length ξ for the two-point local field correlation function has been calculated analytically at zero field, and shown to be the same as the zero-field correlation length for the spin–spin correlation function.

The picture is different for the opposite situation of a field-dominated behaviour ($kT = 0$) of the correlation length. Our exponent of $\alpha = 1.877$ is clearly smaller than the exponent $\alpha_D = 2$ for the spin–spin correlation length [5]. The choice of a new exponent is sustained by the smallness of the errors for $1/\xi$ (see table 1). Therefore statistical errors can be ruled out as the reason for the difference. $\alpha = 2$ gives a distinctly inferior fit to the numerical data compared to $\alpha = 1.877$ (see figure 3).

The different features of the fluctuation variables s_n and x_n must cause the difference in the exponents for the power law divergence. From the mapping, x_n has a distribution for small *non-zero* temperatures which is characterized by a multifractal, while multifractal features have not so far been identified in the distribution for s_n. For *zero* temperatures, x_n has a distribution which consists of a countable number of δ-functions [13]. During the transition from non-zero to zero temperatures x_n undergoes a qualitative change in its distribution, whereas the field-controlled behaviour in the correlation length does not show any change (cf the rightmost part of figure 4 and (17)).

When multifractal features occur, simple relationships (such as gap scaling) between the scaling behaviour of different correlation functions would not be expected [7], and this may be the reason why $\alpha \neq 2$. This question is worthy of further consideration.

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